Math 565: Functional Analysis
Lecture 23

Matrix representation of bdd linear transformations

Let H_1, H_2 be Hilbert spaces with ON bases $(f_j)_{j\in J}$ and $(e_i)_{i\in I}$, resp. Fix $T \in B(H_1, H_2)$. For each $x \in H_1$, we have $x = \sum_{j\in J} \langle x, f_j\rangle f_j$, so linearity and continuity, $Tx = \sum_{j\in J} \langle x, f_j\rangle Tf_j$, so $(Tf_j)_{j\in J}$ encodes T. We further represent $Tf_j = \sum_{i\in I} \langle Tf_j, e_i\rangle e_i$ and denote

$$T_{ij} := \langle Tf_j, e_i\rangle,$$

hence T is encoded by the matrix $(T_{ij})_{i\in I, j\in J}$, which we call the **matrix representation of T** (in the bases $\{f_j\}_{j\in J}$, $\{e_i\}_{i\in I}$).

Compact operators and their spectral theory

Compact operators.

Def. A set Y in a top space X is called **precompact** if it is contained in a compact set (which is just $\overline{Y}$ if X is Hausdorff).

Prop. For a metric space X and $Y \subseteq X$, TFAE:

(1) Y is precompact (i.e. has compact closure).

(2) Y is sequentially precompact, i.e. every sequence has a convergent (in X) subsequence.

(3) Y is totally bdd, i.e. $\forall \varepsilon > 0$ $\exists$ finite cover of Y with set of diam $< \varepsilon$. (♡ intrinsic)

Def. For Banach spaces X, Y, a transf. $T \in B(X,Y)$ is called **compact** if it maps bdd sets to precompact ($\Leftrightarrow$ totall bdd) sets. Equiv, $T(B_1^X)$ is totally bdd. Denote their set by $B_0(X,Y)$.

Example. A transf. $T \in B(X,Y)$ is called **finite rank** if $T(X)$ is finite dimensional. Finite rank transformations are compact since bdd sets in a finite dim space (i.e. $\mathbb{C}^n$) are totally bdd.

Prop. For Banach spaces X, Y, the set $B_c(X,Y)$ is norm-closed in $B(X,Y)$.

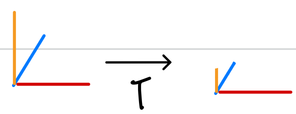
Proof. Let $(T_n) \subseteq B_c(X,Y)$ and suppose $T_n \to T \in B(X,Y)$. Let $\varepsilon > 0$ and take n large enough so that $\|T - T_n\| < \varepsilon/3$. Let $\mathcal{P}$ be a finite cover of $T_n(B_1^X)$ by sets of diam $< \varepsilon/3$, so $\mathcal{Q} := \{T_n^{-1}(P) : P \in \mathcal{P}\}$ is a finite cover of B_1^X. But then $T(\mathcal{Q}) := \{T(Q) : Q \in \mathcal{Q}\}$ is a finite cover of $T(B_1^X)$ and diam $T(Q) < \varepsilon$ because: $\forall x, x' \in Q$

$$\|Tx - Tx'\| \leq \|Tx - T_n x\| + \|T_n x - T_n x'\| + \|T_n x' - Tx'\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.$$

□

Examples. (a) Norm-limits of finite rank transformations are compact.

(b) Let H be a Hilbert space with an ON basis $\{e_i\}_{i\in I}$. Let $\lambda \in \ell^\infty(I)$ and define $T \in B(H)$ by $Tx := \sum_{i\in I} \langle x, e_i\rangle \lambda(i) e_i$, so the e_i are the eigenvectors of T with eigenvalue λ_i. This indeed defines a bdd linear operator because $\sum_{i\in I} |\langle x, e_i\rangle|^2 |\lambda(i)|^2 \leq \|\lambda\|_\infty \|x\|^2$, so $\|T\| \leq \|\lambda\|_\infty$. In fact, $\|T\| = \|\lambda_\infty\|$ because $\|\lambda_\infty\| = \sup_{i\in I} \|Te_i\|$.



Prop. T is compact $\iff$ $\lim_{i\to\infty} \lambda(i) = 0$ (along Fréchet filter, i.e. $\forall \varepsilon > 0$, $I_\varepsilon := \{i \in I : |\lambda(i)| \geq \varepsilon\}$ is finite).

Proof. $\Rightarrow$. If I_ε is infinite, then $\{Te_i : i \in I_\varepsilon\}$ is an infinite set of vectors with pairwise distance $\|Te_i - Te_j\|^2 = \|\lambda(i) e_i - \lambda(j) e_j\|^2 = |\lambda(i)|^2 + |\lambda(j)|^2 \geq 2\varepsilon^2$, so not totally bdd.

$\Leftarrow$. $\lim_{i\to\infty} \lambda(i) = 0$ implies that $I_0 := \{i \in I : \lambda(i) \neq 0\}$ is ctbl, so might as well assume $I = \mathbb{N}$.

Then let $T_n x := \sum_{i<n} \langle x, e_i\rangle \lambda(i) e_i$, so $\|Tx - T_n x\|^2 = \|\sum_{i\geq n} \langle x, e_i\rangle \lambda(i) e_i\|^2 \overset{\text{Parseval}}{=} \sum_{i\geq n} |\langle x, e_i\rangle|^2 |\lambda(i)|^2$

$\leq \sup_{i\geq n} |\lambda(i)|^2 \cdot \|x\|^2$, so

$\|T - T_n\| \leq \sup_{i\geq n} |\lambda(i)| \to 0$ as $n \to \infty$.

But the T_n are finite rank, hence compact, so T is compact. □

Cor. If $T \in B(H)$ is compact and $\{e_i\}_{i \in I}$ is an ON basis for H of eigenvectors of T with eigenvalues λ_i, then $\lim_{i \to \infty} \lambda_i = 0$, in particular, each eigenspace E_{λ_i} is finite dimensional.

Theorem. For Hilbert spaces H_1, H_2, every compact $T \in B(H_1, H_2)$ is a norm-limit of finite rank transformations.

Proof. HW

Cor. If $T \in B(H)$ is compact then T^* is compact as well.

Proof. HW

Obs. If $S, T \in B(H)$ and T is compact, then $S \circ T$ and $T \circ S$ are compact.

Proof. $S \circ T$ is compact because T maps bdd to precompact and S, being continuous, maps precompact to precompact. $T \circ S$ is compact because S maps bdd to bdd and T maps bdd to precompact. □

Remark. $B(H)$ is a C^*-algebra and $B_c(H)$ is a norm-closed two-sided $*$-ideal in $B(H)$.

We now give an important class of examples of compact operators.

Hilbert-Schmidt operators.

Def. Let H be a Hilbert space and $\{e_i\}_{i \in I}$ be an ON basis for it. Call $T \in B(H)$ a Hilbert-Schmidt operator if

$$\|T\|_{HS} := \|T\|_2 := \Big(\sum_{i,j \in I} |T_{ij}|^2\Big)^{\frac{1}{2}} < \infty, \quad \text{where } T_{ij} := \langle Te_j, e_i \rangle.$$

$\|T\|_{HS} = \|T\|_2$ is called the Hilbert-Schmidt norm of T.

Obs. $\|T\|_2^2 = \sum_{j \in I} \|Te_j\|^2$.

Proof. $\|Te_j\|^2 = \sum_{i\in I} |\langle Te_j, e_i\rangle|^2 = \sum_{i\in I} |T_{ij}|^2$ by Parseval, so $\sum_{j\in I} \|Te_j\|^2 = \sum_{i,j\in I} |T_{ij}|^2 = \|T\|_2^2$. $\square$

Obs. For $T\in B(H)$, $\|T\|_2 = \|T^*\|_2$.

Proof. Because $T^*_{ij} = \langle T^* e_j, e_i\rangle = \langle e_j, Te_i\rangle = \overline{\langle Te_i, e_j\rangle} = \overline{T_{ji}}$. $\square$

Prop. The definition of $\|T\|_2$ is independent of the choice of an ON basis.

Proof. Fix two ON bases $\{e_i\}_{i\in I}$ and $\{f_j\}_{j\in J}$. Then by Parseval applied to $\|Te_i\|^2$ wrt $\{f_j\}_{j\in J}$,

$$\|T\|_{2,\{e_i\}} := \sum_{i\in I} \|Te_i\|^2 = \sum_{i\in I}\sum_{j\in J} |\langle Te_i, f_j\rangle|^2 = \sum_{j\in J}\sum_{i\in I} |\langle T^* f_j, e_i\rangle|^2$$

$$(\text{Parseval wrt } \{e_i\}_{i\in I}) = \sum_{j\in J} \|T^* f_j\| = \|T^*\|_{2,\{f_j\}} =: \|T\|_{2,\{f_j\}}.$$ $\square$

Prop. For any linear $T: H\to H$, $\|T\| \le \|T\|_2$.

Proof. Fix an ON basis $\{e_i\}_{i\in I}$. Then $\|Tx\|^2 = \sum_{j\in I} |\langle Tx, e_j\rangle|^2 = \sum_{j\in I} |\langle x, T^* e_j\rangle|^2 \overset{CBS}{\le} \|x\|^2 \sum_{j\in J} \|T^* e_j\|^2$
$= \|x\|^2 \|T^*\|_2^2 = \|x\|^2 \|T\|_2^2$. $\square$

Remark. If $T\in B(H)$ is Hilbert-Schmidt, then there is a separable closed subspace $M\le H$ which is T-invariant (i.e. $T(M)\le M$) and $T|_{M^\perp} \equiv 0$, so restricting T to M, we might as well assume H is separable. Indeed, fixing an ON basis $\{e_i\}_{i\in I}$, $\|T\|_2 < \infty$ implies that $T_{ij} \ne 0$ only for ctbly many pairs $(i,j)\in I^2$, so $I_0 := \{i,j\in I : T_{ij}\ne 0\}$ is ctbl and $M := \overline{\text{span}}\{e_i\}_{i\in I_0}$ is as desired.

Prop. For any Hilbert space H, Hilbert-Schmidt operators on H are compact.

Proof. By the last remark, we may assume WLOG that H has a ctbl ON basis $\{e_n\}_{n\in\mathbb{N}}$. For each $n\in\mathbb{N}$, define $T_n(e_j) := \begin{cases} T(e_j) & \text{if } i<n \\ 0 & \text{o.w.}\end{cases}$. Then $\|T - T_n\|_2^2 = \sum_{j\ge n} \|Te_j\|^2 \to 0$ because it is the tail of a convergent series. Thus, $\|T-T_n\| \le \|T-T_n\|_2 \to 0$, so T is a limit of finite rank operators, hence T is compact. $\square$

Example (L^2 kernels). Let (X,μ) be a σ-finite measure space. For each $K\in L^2(X\times X, \mu\times\mu)$, the operator

$$T_K : L^2(X,\mu) \to L^2(X,\mu) \text{ defined by } T_K f(x) := \langle K_x, \bar{f}\rangle = \int K(x,y) f(y)\, d\mu(y)$$

is Hilbert-Schmidt with $\|T_K\|_2 = \|K\|_2$. (By K_x we mean the fiber-function $y\mapsto K(x,y)$.)

Proof. By Fubini, $\|K\|_2^2 = \int |K|^2 \, d\mu\times\mu = \iint |K(x,y)|^2 \, dy\, dx = \int \|K_x\|_2^2 \, dx$, so $\|K_x\|_2 < \infty$ for a.e. $x \in X$, i.e. $K_x \in L^2(X,\mu)$ for a.e. $x \in X$. Thus, $T_K f(x) = \langle K_x, \bar{f} \rangle$ is defined at a.e. $x \in X$. Moreover:

$$\|T_K f\|_2^2 = \int |\langle K_x, \bar{f}\rangle|^2 \, d\mu(x) \leq \int \|K_x\|_2^2 \|f\|_2^2 \, d\mu = \|f\|_2^2 \int \|K_x\|^2 \, d\mu(x) = \|f\|_2^2 \, \|K\|_2^2,$$

so $\|T_K\| \leq \|K\|$. To show $\|T_K\|_2 = \|K\|_2$, fix an ON basis $\{e_i\}_{i\in I} \subseteq L^2(X,\mu)$ and compute:

$$\|T_K\|_2^2 = \sum_{i\in I} \|T_K e_j\|_2^2 = \sum_{i\in I} \int |\langle K_x, \bar{e}_j\rangle|^2 \, d\mu(x) = \int \sum_{i\in I} |\langle K_x, \bar{e}_j\rangle|^2 \, d\mu(x) = \int \|K_x\|_2^2 \, d\mu(x) = \|K\|_2^2. \quad \square$$

Parseval, since $\{\bar{e}_i\}$ is an ON basis